JOURNAL OF NUMBER THEORY 8, 12–39 (1976)

On Lattice Points in Multidimensional Ellipsoids: Problem of Centers

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Communicated by A. Woods

Received December 31, 1972

The paper determines the exact order of the lattice remainder term for an integral ellipsoid in terms of the arithmetic character of its center.

1. INTRODUCTION

Let $r \geqslant 2$ be a positive integer and let

$$Q(u) = Q(u_j) = \sum_{l,j=1}^{r} a_{lj} u_l u_j \tag{1}$$

be a positive definite quadratic form with a symmetric coefficient matrix of determinant D. Let further $\alpha_1, \alpha_2, \ldots, \alpha_r, b_1, b_2, \ldots, b_r$ be real numbers. For $x \geqslant 0$, denote by $A(x)$ the sum

$$\sum \exp\left(2\pi i \sum_{j=1}^{r} \alpha_j u_j\right), \tag{2}$$

where the summation runs over all r-tuples $u = (u_1, u_2, \ldots, u_r)$ of integers such that $Q(u_j + b_j) \leqslant x$. Let us put

$$V(x) = \pi^{r/2} x^{r/2} \delta / (D)^{1/2}\, \Gamma((r/2) + 1), \tag{3}$$

where $\delta = 1$ if all α_j are integers and $\delta = 0$ otherwise. Let us remark that the function $A(x)$ is usually defined by (2), where the summation runs over all r-tuples $u = (u_1, u_2, \ldots, u_r)$ of real numbers such that $Q(u) \leqslant x$ and $u_j \equiv b_j \pmod{M_j}$, where M_j are given positive real num-

* This paper was written during the author's stay at the University of Illinois, Urbana, and at the Ohio State University, Columbus.

bers, $j = 1, 2, \ldots, r$. It is easy to see that we obtain this new function from our function upon replacing a_{lj} by $M_l M_j a_{lj}$, b_j by $M_j b_j$, α_j by $\alpha_j M_j$, $j = 1, 2, \ldots, r$, and multiplying by $\exp(2\pi i \sum_{j=1}^{r} \alpha_j b_j)$.

The basic question in the theory of lattice points in ellipsoids, as proposed by Landau, is the investigation of the best possible O- and Ω-estimates of the "lattice remainder term"

$$P(x) = A(x) - V(x). \tag{4}$$

In 1968, Jarník, in his last paper [7] on lattice point theory, proposed the more general question of finding the "exact order" of the function

$$P_\rho(x) = \frac{1}{\Gamma(\rho)} \int_0^x P(t)(x - t)^{\rho - 1}\, dt, \qquad \rho > 0 \tag{5}$$

(we put $P_0(x) = P(x)$). It was shown that in many important special cases the above-mentioned "exact order" of the function $P_\rho(x)$, i.e., the value

$$f_\rho = \limsup_{x \to +\infty} \frac{\log |P_\rho(x)|}{\log x}, \tag{6}$$

depends substantially on the arithmetic properties of numbers a_{lj}, b_j and α_j, provided ρ is not too large. Let us recall some results that are most interesting from our point of view.

For given real numbers $\beta_1, \beta_2, \ldots, \beta_n$ we define $\gamma = \gamma(\beta_1, \beta_2, \ldots, \beta_n)$ as the supremum of all positive β, for which the system of inequalities

$$|q\beta_j - p_j| < q^{-\beta}, \qquad j = 1, 2, \ldots, n,$$

has infinitely many solutions in integers $p_1, p_2, \ldots, p_n$ and q.

Theorem A. *Let the form* (1) *have the following "almost diagonal" form*

$$Q(u) = a_1 Q_1(u_1, u_2, \ldots, u_{r_1}) + a_2 Q_2(u_{r_1+1}, u_{r_1+2}, \ldots, u_{r_1+r_2}) + \cdots$$
$$+ a_\sigma Q_\sigma(u_{r-r_\sigma+1}, \ldots, u_r),$$

where a_j are positive real numbers, Q_j positive definite quadratic forms with integral coefficients, r_j and σ positive integers, $j = 1, 2, \ldots, \sigma$, and $r_1 + r_2 + \cdots + r_\sigma = r$. Let $\alpha_j = b_j = 0$, $j = 1, 2, \ldots, r$ and $\gamma = \gamma(1, a_2/a_1, a_3/a_1, \ldots, a_\sigma/a_1)$.

Then[1]

$$f_\rho = (r/2) - 1 - ((\rho + 1)/\gamma)$$

provided $r_j \geqslant 2(\rho + 1)(\gamma + 1)/\gamma, j = 1, 2,..., \sigma, 0 \leqslant \rho < (r/2) - 2.$

THEOREM B. *Let the form* (1) *have integral coefficients and* $b_1 = b_2 = \cdots = b_r = 0$. *We put* $\gamma = \gamma(\alpha_1, \alpha_2, ..., \alpha_r)$. *Then*

$$f_\rho = (r/2) - 1 - (1/2(\gamma + 1))((r/2) - 1 - \rho)$$

provided $0 \leqslant \rho \leqslant (r/2) - 2 - (1/\gamma)$.

For the proof of this theorem see [17]. For a large value of ρ, the value of f_ρ is known in the general case.[2]

THEOREM C. *Let* $A(x) \not\equiv 0$. *Then*

$$P_\rho(x) = \Omega(x^{(r-1)/4+\rho/2})$$

and for $\rho > r/2 - 1/2$ *we even have*

$$P_\rho(x) = O(x^{(r-1)/4+\rho/2}).$$

The aim of the present paper is to investigate the dependence of f_ρ on the "center" $b = (b_1, b_2, ..., b_r)$. We shall suppose in the sequel that the form (1) has integral coefficients and $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$. Besides Landau's old result $(r - 1)/4 \leqslant f_0 \leqslant (r/2) - (r/(r + 1))$ (see [9, pp. 11–84]), only some partial results were known in this case:

It was proved by Landau, Walfisz, and Jarník (see [9, pp. 157–162]), that

$$f_0 = (r/2) - 1$$

provided $r \geqslant 4$ and the center $(b_1, b_2, ..., b_r)$ is rational. Using the same method as in [7] it is possible to show that for these b_j's we have

$$f_\rho = (r/2) - 1$$

provided $0 \leqslant \rho \leqslant (r/2) - 2$ and

$$\max\left(\frac{r-1}{4} + \frac{\rho}{2}, \frac{r}{2} - 1\right) \leqslant f_\rho \leqslant \min\left(\frac{r}{4} + \frac{\rho}{2}, \frac{r}{2} - \frac{r}{r + 1 - 2\rho} + \rho\right)$$

provided $(r/2) - 2 \leqslant \rho \leqslant (r/2) - (1/2)$.

[1] For $\gamma = +\infty$, we define the value of all expressions involving γ by their limit for $\gamma \to +\infty$. The proof of this theorem is contained in [1, 3].

[2] Jarník proved this result in [7] under the assumption $\alpha_j = b_j = 0, j = 1, 2,..., r$. It is easy to see that his proof gives also the result in Theorem C.

In 1968, the author proved (see [16]) a certain "duality theorem," and using his own results of a similar kind as in Theorem B he proved for example that

$$P(x) = o(x^{r/2-r/(r+1)})$$

if $r > 4$ and at least one of the numbers $b_1, b_2, \ldots, b_r$ is irrational and that

$$P(x) = O(x^{r/3+\epsilon})$$

for every $\epsilon > 0$ and almost all systems $b_1, b_2, \ldots, b_r$ (in the sense of Lebesgue measure) provided $r > 5$.

A very surprising result was published by Kendall in 1948 (see [8]). We shall present its generalization, which was proved in a different way in [15]:

$$\int_0^1 \int_0^1 \cdots \int_0^1 | P(x)|^2 \, db_1 \, db_2 \cdots db_r = \begin{cases} O(x^{r/2-1/2}), \\ \Omega(x^{r/2-1/2}), \end{cases}$$

for all Q and all systems $\alpha_1, \alpha_2, \ldots, \alpha_r$ of real numbers.

Finally, in 1970 B. Diviš, by a clever modification of Jarník's old Ω-method, proved (unpublished till now but see [19]) that

$$P(x) = \Omega(x^{(r/2)-1-(1/2\beta)})$$

provided the system of inequalities

$$| qb_j - p_j | < q^{-\beta}$$

has infinitely many solutions in integers $p_1, p_2, \ldots, p_r$ and q.

In this paper we shall prove, in addition to other results, the following

MAIN THEOREM. *Let the form* (1) *have integral coefficients and* $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$. *Let* $\gamma = \gamma(b_1, b_2, \ldots, b_r)$. *Then*

$$f_\rho = (r/2) - 1 - ((\rho + 1)/2\gamma)$$

provided $0 \leqslant \rho \leqslant (r/2) - 2 - ((r-2)/2(\gamma + 1))$.

Let

$$M_\rho(x) = \int_0^x | P_\rho(t)|^2 \, dt.$$

Then

$$\limsup_{x\to+\infty} \log M_\rho(x)/\log x = r - 1 - ((\rho + 1)/\gamma)$$

provided $0 \leqslant \rho \leqslant (r/2) - \frac{3}{2} - ((r-1)/2(\gamma+1))$. *If* $\rho > (r/2) - \frac{3}{2} - ((r-1)/2(\gamma+1))$, $\rho \geqslant 0$, *then*

$$x^{(r/2)+\rho+(1/2)} \ll M_\rho \ll x^{(r/2)+\rho+(1/2)} \log^\kappa x,$$

where $\kappa = 1$ *for* $\rho = (r/2) - \frac{3}{2} \geqslant 0$ *and* $\kappa = 0$ *otherwise.*[3]

The general method used for O-estimates is due to Jarník (cf., e.g., [5, 6]) and the author (cf. [10, 12, 14]). For Ω-estimates we shall use the method from [10]. Our transformation formula for the corresponding theta function is a special case of a more general formula from [4, 21]. For O-estimates we shall use the estimates of certain number theoretical sums, which have been proved in [18]. The proof of these estimates makes essential use of a lemma belonging to the theory of diophantine approximations that was proved in [1]. Some new results have been published in [21].

2. Notations and Auxiliary Theorems

We shall use throughout the paper the following conventions and notations.

By Q we shall always mean the form (1) with integral coefficients; the functions $P_\rho(x)$, $M_\rho(x)$ and the value f_ρ and $\gamma = \gamma(b_1, b_2, \ldots, b_r)$ are defined as in the Introduction.

The letter c denotes (in general, different) positive constants, depending only on ρ, Q, and $b_1, b_2, \ldots, b_r$. The number x will be sufficiently large, $x > c$. Instead of $|A| \leqslant cB$, we shall write only $A \ll B$; if, in addition, $B \ll A$, we write $A \asymp B$. The letters h and m_j denote integers, the letters k and n nonnegative integers, $k > 0$. If h and k occur simultaneously, it is always $(h, k) = 1$ (the same for h_1 and k_1, etc.). By the symbol $\sum_{h,k}$ we mean the summation over all $h, k, h > 0, k \leqslant (x)^{1/2}$. If not stated explicitly otherwise, by an integral we always mean the (absolutely convergent) Lebesgue integral. For a real a, let

$$\int_{(a)} f(s)\, ds = i \int_{-\infty}^{\infty} f(a + it)\, dt$$

and (for $-\infty \leqslant a \leqslant b \leqslant \infty$ and $J = [a, b)$)

$$\int_J f(s)\, dt = \int_a^b f(x^{-1} + it)\, dt,$$

provided the integrals on the right-hand sides exist.

[3] One can find the results about the function $M_\rho(x)$ in the cases dealt with in Theorems A and B in [2, 6, 12, 13, 20].

For a real number t, $\langle t \rangle$ will denote the distance of t to the nearest integer. Let $\bar{Q}$ be the quadratic form conjugate to Q and let

$$R_h = \min_{m_1, m_2, \ldots, m_r \text{ int.}} \bar{Q}\left(m_j - 2h \sum_{l=1}^{r} a_{jl} b_l\right)$$

and

$$P_h = \max_{j=1,2,\ldots,r} \langle h b_j \rangle.$$

We prove the following easy lemma.

LEMMA 1. *Let α and β be nonnegative numbers. Then the inequality*

$$R_h \ll h^{-2\beta} \log^{-2\alpha} h$$

has infinitely many solutions in positive integers h if and only if the same assertion holds for the inequality

$$P_h \ll h^{-\beta} \log^{-\alpha} h.$$

Proof. Let

$$\tilde{P}_h = \max_{j=1,2,\ldots,r} \left\langle 2h \sum_{l=1}^{r} a_{jl} b_l \right\rangle.$$

For any system of real numbers $u_1, u_2, \ldots, u_r$ we have

$$\bar{Q}(u_j) \asymp \sum_{j=1}^{r} u_j^2 \asymp (\max_{j=1,2,\ldots,r} | u_j |)^2.$$

From this relation we obtain immediately

$$R_h \asymp \tilde{P}_h^2. \tag{7}$$

Thus, it will be sufficient to prove the assertion formulated above for the inequalities

$$P_h \ll h^{-\beta} \log^{-\alpha} h \qquad \text{and} \qquad \tilde{P}_h \ll h^{-\beta} \log^{-\alpha} h.$$

Let, for certain integers $m_1, m_2, \ldots, m_r$,

$$2h \sum_{l=1}^{r} a_{jl} b_l = m_j + \epsilon_j .$$

$j = 1, 2, \ldots, r$. Then (by Cramer's rule)

$$2hDb_l = \sum_{j=1}^{r} A_{lj}(m_j + \epsilon_j), \qquad l = 1, 2, \ldots, r,$$

where A_{lj} are integers, depending only on Q. Thus

$$2hDb_l - \sum_{j=1}^{r} A_{lj}m_j = \sum_{j=1}^{r} A_{lj}\epsilon_j\,, \qquad l = 1, 2, \ldots, r.$$

Conversely, for certain integers $p_1, p_2, \ldots, p_r$, let

$$hb_j = p_j + \delta_j\,, \qquad j = 1, 2, \ldots, r.$$

Then

$$2h \sum_{l=1}^{r} a_{jl}b_l - \sum_{l=1}^{r} a_{jl}p_l = \sum_{l=1}^{r} a_{jl}\delta_l\,, \qquad j = 1, 2, \ldots, r.$$

From these considerations it now follows easily that

$$P_{2hD} \leqslant \tilde{P}_h \qquad \text{and} \qquad \tilde{P}_h \leqslant P_h \tag{8}$$

and thus also our assertions.

Let $\lambda_0 = 0$ and $0 < \lambda_1 < \lambda_2 < \cdots$ be the sequence of all positive numbers of the form $Q(m_j + b_j)$ with integers $m_1, m_2, \ldots, m_r$, and let a_n denote the number of solutions of the equation

$$Q(m_j + b_j) = \lambda_n$$

in integers $m_1, m_2, \ldots, m_r$. Obviously,

$$A(x) = \sum_{\lambda_n \leqslant x} a_n$$

and

$$P_\rho(x) = \frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leqslant x} a_n(x - \lambda_n)^\rho - \frac{\pi^{r/2} x^{(r/2)+\rho}}{(D)^{1/2}\, \Gamma((r/2) + \rho + 1)} \tag{9}$$

for $\rho \geqslant 0$. For a complex s with $\mathrm{Re}(s) > 0$, we define the function $\Theta(s)$ (i.e., the theta-function corresponding to our problem) by the relation

$$\Theta(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}. \tag{10}$$

This function is obviously holomorphic in the whole half-plane $\mathrm{Re}(s) > 0$.

Let us put

$$F(s) = \Theta(s) - \frac{\pi^{r/2}}{D^{1/2}s^{r/2}} \quad \text{and} \quad G(s) = \overline{F(\bar{s})}. \tag{11}$$

(For a positive β, we mean by s^β that branch of the analytic function s^β which is positive for positive values of s.) From Hankel's formula it follows that

$$P_\rho(x) = \frac{1}{2\pi i} \int_{(a)} \frac{e^{xs}F(s)}{s^{\rho+1}}\, ds \tag{12}$$

provided $a > 0$ and $\rho > 0$. In Lemma 1 of [12, p. 58] it has been proved that for $\rho \geqslant 0$, $n = 1, 2, \ldots$,

$$M_{\rho,n}(x) = -\frac{1}{4\pi^2} \int_{(a)} \left(\int_{(a)} \frac{F(s)\, G(s')\, e^{x(s+s')}}{s^{\rho+1} s'^{\rho+1} (s + s')^n}\, ds \right) ds', \tag{13}$$

where $M_{\rho,1}(x) = M_\rho(x)$ and

$$M_{\rho,n+1}(x) = \int_0^\infty M_{\rho,n}(t)\, dt, \qquad n = 1, 2, \ldots .$$

These expressions will be the starting point of our investigations. From (12) and (13) we see that it will be useful to put $a = 1/x$, and thus we need good estimates of the functions (11) near the imaginary axis. Next, we prove the following transformation property of the function $\Theta(s)$.

LEMMA 2. *Let s be a complex number*, $\mathrm{Re}(s) > 0$. *Then*

$$\Theta(s) = \frac{\pi^{r/2}}{(s - (2\pi i h/k))^{r/2}\, k^r D^{1/2}} \times \sum_{m_1, m_2, \ldots, m_r} S_{h,k,(m)} \exp\left(- \frac{\pi^2 \bar{Q}(m_j - 2h \sum_{l=1}^r a_{jl} b_l)}{k^2 (s - (2\pi i h/k))} \right), \tag{14}$$

where

$$\begin{aligned} S_{h,k,(m)} &= S_{h,k,(m_1, m_2, \ldots, m_r)} \\ &= \exp\left(\frac{2\pi i h}{k} Q(b_j) - \frac{2\pi i}{k} \sum_{j=1}^r m_j b_j \right) \\ &\quad \times \sum_{a_1, a_2, \ldots, a_r = 1}^{k} \exp\left(- \frac{2\pi i h}{k} Q(a_j) - \frac{2\pi i}{k} \sum_{j=1}^r m_j a_j \right). \end{aligned} \tag{15}$$

Proof. Let $s = s' + (2\pi ih/k)$. Then

$$\Theta(s) = \sum_{n_1,n_2,\ldots,n_r} \sum_{a_1,a_2,\ldots,a_r=1}^{k} \exp\left(-\left(s' + \frac{2\pi ih}{k}\right) Q(n_jk + a_j + b_j)\right)$$

$$= \sum_{a_1,a_2,\ldots,a_r=1}^{k} \exp\left(-\frac{2\pi ih}{k} Q(a_j + b_j)\right)$$

$$\times \sum_{n_1,n_2,\ldots,n_r} \exp\left(-s'k^2Q\left(n_j + \frac{a_j + b_j}{k}\right) - 4\pi ih \sum_{j,l=1}^{r} n_ja_{jl}b_l\right).$$

Now, we use the well-known transformation formula for theta function (see, e.g., [9, p. 239], Theorem 3, where we write $s'(k^2/\pi)$ instead of s and put $\gamma_j = (a_j + b_j)/k$ and $\delta_j = -2h \sum_{l=1}^{r} a_{jl}b_l$, $j = 1, 2,\ldots, r$). We obtain

$$\Theta(s) = \frac{\pi^{r/2}}{s'^{r/2}k^rD^{1/2}} \sum_{a_1,a_2,\ldots,a_r=1}^{k} \exp\left(-\frac{2\pi ih}{k} Q(a_j + b_j)\right.$$

$$\left. + 2\pi i \sum_{j=1}^{r} \frac{a_j + b_j}{k} 2h \sum_{l=1}^{r} a_{jl}b_l\right)$$

$$\cdot \sum_{n_1,n_2,\ldots,n_r} \exp\left(-\frac{\pi^2}{k^2s'} \bar{Q}\left(n_j - 2h \sum_{l=1}^{r} a_{jl}b_l\right)\right.$$

$$\left. - 2\pi i \sum_{j=1}^{r} m_j \frac{a_j + b_j}{k}\right),$$

i.e., we obtain the relation (14).

For the sums $S_{h,k,(m)}$ we have the estimate

$$S_{h,k,(m)} \ll k^{r/2}. \tag{16}$$

For the proof of (16) it is sufficient to prove the relation

$$\sum_{a_1,a_2,\ldots,a_r=1}^{k} \exp\left(-\frac{2\pi ih}{k} Q(a_j) - \frac{2\pi i}{k} \sum_{j=1}^{r} m_ja_j\right) \ll k^{r/2}.$$

For a proof see [10, p. 376, Lemma 2].

From Lemma 2 we obtain now some very important information about the behavior of the function $\Theta(s)$ near the imaginary axis. We will formulate this property in a form that will be useful for our purposes. First, we define the numbers $S_{h,k}$ as follows. Let the relation

$$R_h = \bar{Q}\left(m_j - 2h \sum_{l=1}^{r} a_{jl}b_l\right) \tag{17}$$

be fulfilled for just one system of integers $m_1, m_2, \ldots, m_r$. We then put

$$S_{h,k} = S_{h,k,(m_1,m_2,\ldots,m_r)}\,; \tag{18}$$

otherwise, we choose one of the systems $m_1, m_2, \ldots, m_r$ for which (17) holds, and define the value $S_{h,k}$ again by (18). From (7) it is easy to see that there exists a constant $c_1 = c$ such that if $R_h < c_1$, then (17) holds for just one system $m_1, m_2, \ldots, m_r$. We shall see that this arbitrariness of choice does not have any influence on our results.

LEMMA 3. *Let* $s = (1/x) + it$ *and* $|\, t - (2\pi h/k)| \ll 1/k(x)^{1/2}$. *Then*

$$F(s)/s^{\rho+1} \ll x^{r/4+\rho/2+1/2}, \tag{19}$$

provided $h = 0$ (*and thus* $k = 1$) *and*

$$\Theta(s) \ll \left(\frac{x}{k}\right)^{r/2} \frac{\exp\left(-\dfrac{cR_h x}{k^2(1 + x^2\,|\, t - (2\pi h/k)|^2)}\right)}{(1 + x^2\,|\, t - (2\pi h/k)|^2)^{r/4}} \tag{20}$$

for $h \neq 0$.

The proof can be obtained from (14) and (16) in the same way as the proof of Lemma 3 in [12, pp. 61–62].

LEMMA 4. *Let* $s = \sigma + (2\pi i h/k)$ *with* $\sigma > 0$. *Then*

$$\Theta(s) - \frac{\pi^{r/2} S_{h,k} e^{-\pi^2 R_h/k^2\sigma}}{D^{1/2}k^r\sigma^{r/2}} \ll \frac{e^{-c/k^2\sigma}}{\sigma^{r/2}k^{r/2}}. \tag{21}$$

The proof can be obtained from (14) and (16) in the same way as the proof of Lemma 2 in [11, p. 226].

We conclude this paragraph by two lemmas of a technical character.

LEMMA 5. *Let* $\beta = c$ *and* $T \geqslant 0$. *Then*

$$\int_0^{c/k(x)^{1/2}} \frac{\exp\left(-\dfrac{cTx}{k^2(1 + x^2u^2)}\right)}{(1 + x^2u^2)^{\beta}}\, du$$

$$\ll \begin{cases} (k^{2\beta-1}/x^{\beta+(1/2)}) \min^{\beta-(1/2)}(x/k^2,\, 1/T) & \text{for } \beta > \tfrac{1}{2}, \\ (\log x)/x & \text{for } \beta = \tfrac{1}{2}.^{4} \end{cases}$$

[4] We put $\min(A, 1/T) = A$ for $T = 0$.

Proof. It is sufficient to estimate the integral

$$I_1 = \int_0^{c(x)^{1/2}/k} \frac{e^{-T_1/(1+u^2)}}{(1+u^2)^\beta}\, du,$$

where $T_1 \geqslant 0$. If $\beta = \frac{1}{2}$, then

$$I_1 \ll \int_0^{c(x)^{1/2}/k} \frac{du}{1+u} \ll \log x.$$

For $\beta > \frac{1}{2}$, we obtain

$$I_1 = \int_0^{c(x)^{1/2}/k} \frac{du}{(1+u^2)^\beta} \ll 1$$

for $T_1 \geqslant 0$ and

$$I_1 \ll \frac{1}{T_1^\beta} \int_0^\infty \left(\frac{T_1}{1+u^2}\right)^\beta e^{-T_1/(1+u^2)}\, du$$

$$\ll \frac{1}{T_1^\beta} \left(\int_0^{(T_1)^{1/2}} du + \int_{(T)^{1/2}}^\infty \frac{T_1^\beta}{u^{2\beta}}\, du\right) \ll T_1^{(1/2)-\beta}$$

for $T_1 > 0$, since $\xi^\beta e^{-\xi} \ll 1$ for $\xi \in [0, \infty)$.

LEMMA 6. *For $n > c$, $t \geqslant w$, and $w \asymp x^{-1/2}$, we have*

$$\int_w^\infty \frac{dt'}{t'^{\rho+1}((1/x) + |t - t'|)^n} \ll \frac{x^{n-1}}{t^{\rho+1}}.$$

Proof. See [12, p. 68, relation (26)].

3. General Formulas for O-Estimates

Let us recall some known properties of the Farey fractions corresponding to $(x)^{1/2}$, i.e., fractions of the form h/k, where $k \leqslant (x)^{1/2}$ (cf. [9, pp. 249–250]). If $h_1/k_1 < h/k < h_2/k_2$ are three consecutive fractions of this type (i.e., between h_1/k_1 and h_2/k_2 lies just one Farey fraction corresponding to $(x)^{1/2}$, namely, h/k), then we denote by $\mathfrak{B}_{h,k}$ the interval

$$\left[2\pi \frac{h+h_1}{k+k_1},\, 2\pi \frac{h+h_2}{k+k_2}\right].$$

We have

$$\mathfrak{B}_{h,k} = \left[2\pi \frac{h}{k} - \frac{\theta_1}{k(x)^{1/2}}, 2\pi \frac{h}{k} + \frac{\theta_2}{k(x)^{1/2}}\right],$$

where $\pi \leqslant \theta_1\theta_2 \leqslant 2\pi$. Thus, for $t \in \mathfrak{B}_{h,k}$ we have

$$| t - (2\pi h/k)| \ll 1/k(x)^{1/2}. \tag{22}$$

The union of all these intervals is the whole real axis. In particular, we have $\mathfrak{B}_{0,1} = [-w, w)$, where $w = 2\pi/(1 + [(x)^{1/2}]) \ll x^{-1/2}$. From (22) it follows easily that if $s = (1/x) + it$, $h \neq 0$ and $t \in \mathfrak{B}_{h,k}$, then

$$| s | \asymp | t | \asymp | h |/k. \tag{23}$$

We use these results in the sequel without any further reference.

First, we find an upper estimate for the function $P_\rho(x)$, $\rho > 0$. We put $a = 1/x$ in (12). According to (19), we obtain ($e^{x((1/x)+it)} \ll 1$)

$$\int_{\mathfrak{B}_{0,1}} \frac{e^{xs}F(s)}{s^{\rho+1}}\, dt \ll x^{r/4+\rho/2+1/2} \int_0^{c/(x)^{1/2}} dt \ll x^{r/4+\rho/2}. \tag{24}$$

Now, it is sufficient to estimate the integral

$$\int_w^\infty (e^{xs}F(s)/s^{\rho+1})\, dt,$$

because the integral

$$\int_{-\infty}^{-w} (e^{xs}F(s)/s^{\rho+1})\, dt$$

has a complex conjugate value. Let

$$I = \int_w^\infty | F(s)/s^{\rho+1} |\, dt. \tag{25}$$

According to (20) and (23), we obtain

$$I \ll \sum_{h,k} (k/h)^{\rho+1} (x/k)^{r/2} \int_0^{c/k(x)^{1/2}} \frac{e^{-cR_h x/k^2(1+x^2u^2)}}{(1 + x^2u^2)^{r/4}}\, du$$

$$+ \sum_{h,k} (k/h)^{r/2+\rho+1} \frac{1}{k(x)^{1/2}}. \tag{26}$$

According to Lemma 5, we obtain from (26)

$$I \ll \sum_{h,k} (k/h)^{\rho+1} (x/k)^{r/2} \frac{k^{r/2-1}}{x^{r/4+1/2}} \min{}^{r/4-1/2}(x/k^2, 1/R_h) \log^{\kappa} x + x^{r/4+\rho/2},$$

where $\kappa = 1$ for $r = 2$ and $\kappa = 0$ otherwise, and thus

$$I \ll x^{r/4-1/2} \log^{\kappa} x \sum_{h,k} (k^{\rho}/h^{\rho+1}) \min{}^{r/4-1/2}(x/k^2, 1/R_h), \tag{27}$$

since

$$x^{r/4-1/2} \sum_{h,k} (k^{\rho}/h^{\rho+1}) \min{}^{r/4-1/2}(x/k^2, 1/R_h) \gg x^{r/4-1/2} \sum_{k \leqslant (x)^{1/2}} k^{\rho} \gg x^{r/4+\rho/2}. \tag{28}$$

From (24), (27), and (28), we obtain

THEOREM 1.

$$P_{\rho}(x) \ll x^{r/4-1/2} \log^{\kappa} x \sum_{h,k} (k^{\rho}/h^{\rho+1}) \min{}^{r/4-1/2}(x/k^2, 1/R_h) \tag{29}$$

for $\rho > 0$, where $\kappa = 1$ for $r = 2$ and $\kappa = 0$ otherwise.

Let us remark that for $\rho > (r/2) - 2$ we have

$$P_{\rho}(x) \ll x^{r/2-1} \log^{\kappa} x \sum_{k \leqslant (x)^{1/2}} k^{\rho-r/2+1} \ll x^{r/4+\rho/2} \log^{\kappa} x, \tag{30}$$

i.e., according to (28), we have the best possible estimate that can be obtained from Theorem 1. By comparison with Theorem C we see that for some values of ρ (especially for $\rho > r/2 - 1/2$) we know better results.

Now, we consider the case $\rho = 0$. In this case we cannot use the relation (12) directly. This relation holds also for $\rho = 0$ but with the function $P'(x) = \frac{1}{2}(P(x + 0) + P(x - 0))$ on the left-hand side and with a conditionally convergent integral on the right-hand side. However, we shall succeed in a different way. Let z be a positive real number, $z \ll 1$. The function $A(x)$ is nonnegative and nondecreasing. Thus

$$(1/z) \int_{x-z}^{x} A(t)\, dt \leqslant A(x) \leqslant (1/z) \int_{x}^{x+z} A(t)\, dt. \tag{31}$$

Now

$$\pm(1/z) \int_{x}^{x\pm z} t^{r/2}\, dt - x^{r/2}$$
$$= \pm(1/z) \int_{x}^{x\pm z} (t^{r/2} - x^{r/2})\, dt \ll (1/z) \int_{0}^{z} x^{r/2-1} u\, du \ll z x^{r/2-1}.$$

Thus,

$$(1/z)\int_{x-z}^{x} P(t)\,dt + O(zx^{r/2-1}) \ll P(x) \ll (1/z)\int_{x}^{x+z} P(t)\,dt + O(zx^{r/2-1}). \tag{32}$$

Now it is sufficient to estimate the integrals

$$(1/z)\int_{x}^{x\pm z} P(t)\,dt = (1/z)(P_1(x\pm z) - P_1(x)).$$

We use the expression (12), where we put in both cases $a = 1/x$. We obtain

$$J = \frac{1}{z}\int_{x}^{x\pm z} P(t)\,dt = \frac{1}{2\pi iz}\int_{(a)} \frac{e^{(x\pm z)s} - e^{xs}}{s^2} F(s)\,ds.$$

For $s = (1/x) + it$ we can use the inequality

$$e^{(x\pm z)s} - e^{xs} \ll |\, e^{\pm zs} - 1 \,| \ll \min(z\,|\,t\,|, 1) + (z/x),$$

and thus by (19)

$$J \ll (1/z)\int_{-\infty}^{\infty} |\, F(s)/s^2 \,| \min(z\,|\,t\,|, 1)\,dt + x^{r/4-1}.$$

We proceed in a similar way as above. First, by (19),

$$(1/z)\int_{\mathfrak{B}_{0,1}} |\, F(s)/s^2 \,| \min(z\,|\,t\,|, 1)\,dt \ll x^{r/4+1}(1/z)\int_{0}^{c/(x)^{1/2}} zt\,dt \ll x^{r/4}. \tag{33}$$

As above, we need only an estimate of

$$J_1 = (1/z)\int_{w}^{\infty} |\, F(s)/s^2 \,| \min(zt, 1)\,dt.$$

From (20) and (23), we obtain

$$J_1 \ll x^{r/4-1/2}\log^\kappa x/z \sum_{h,k} (k/h^2)\min(z(h/k), 1)\min{}^{r/4-1/2}(x/k^2, 1/R_h)$$
$$+ (1/z)\sum_{h,k} (k/h)^{2+r/2}\,(1/k(x)^{1/2})\min(z(h/k), 1). \tag{34}$$

Now

$$\frac{x^{-1/2}}{z}\sum_{k\leqslant x^{1/2}} k^{1+r/2}\sum_{h=1}^{\infty}\frac{\min(z(h/k), 1)}{h^{2+r/2}}$$
$$\ll \frac{x^{-1/2}}{z}\sum_{k\leqslant x^{1/2}} k^{1+r/2}\left(\frac{z}{k}\sum_{h\leqslant k/z}\frac{1}{h^{1+r/2}} + \sum_{h>k/z}\frac{1}{h^{2+r/2}}\right)$$
$$\ll x^{-1/2}\sum_{k\leqslant x^{1/2}} k^{r/2} \ll x^{r/4} \tag{35}$$

and

$$\frac{x^{r/4-1/2}}{z}\sum_{h,k}(k/h^2)\min(z(h/k),1)\min{}^{r/4-1/2}(x/k^2,1/R_h)$$

$$\gg \frac{x^{r/4-1/2}}{z}\sum_{k\leqslant x^{1/2}} k\sum_{h=1}^{\infty}\frac{\min(z(h/k),1)}{h^2}$$

$$=\frac{x^{r/4-1/2}}{z}\sum_{k\leqslant x^{1/2}} k\left(\frac{z}{k}\sum_{h\leqslant k/z}\frac{1}{h}+\sum_{h>k/z}\frac{1}{h^2}\right)$$

$$\gg x^{r/4-1/2}\sum_{k\leqslant x^{1/2}}\log(k/z)\gg x^{r/4}\log(x^{1/2}/z). \tag{36}$$

Summing up, we have proved

THEOREM 2. *Let $0<z\leqslant 1$. Then*

$$P(x)\ll x^{r/4-1/2}\log^{\kappa} x/z\sum_{h,k}(k/h^2)\min(z(h/k),1)$$
$$\times\min{}^{r/4-1/2}(x/k^2,1/R_h)+O(zx^{r/2-1}), \tag{37}$$

where κ is defined as in Theorem 1.

By an argument similar to that following Theorem 1, we can use (36) to show that for $r=2,3,4$ (these are the only values of r satisfying the inequality $0=\rho\geqslant(r/2)-2$) we cannot obtain from (37) a better result than

$$P(x)\ll x^{r/4}\log^{\kappa+1}x$$

(in (37) we put $z=c$). From these considerations we see that our formulas give good results for $0\leqslant\rho<(r/2)-2$ only.

Now we turn to the function $M_\rho(x)$. We shall proceed by an adaptation of the method of [12, Part II]. First, we prove

LEMMA 7. *If $n=c$ is sufficiently large, then*

$$M_{\rho,n}(x)\ll x^{r/2+n-3/2}\sum_{h,k}(k^{2\rho+1}/h^{2\rho+2})\min{}^{r/2-1/2}(x/k^2,1/R_h). \tag{38}$$

From this lemma, we can obtain very easily

THEOREM 3.

$$M_\rho(x)\ll x^{r/2-1/2}\sum_{h,k}(k^{2\rho+1}/h^{2\rho+2})\min{}^{r/2-1/2}(x/k^2,1/R_h). \tag{39}$$

Proof. Both sides of inequality (38) are nondecreasing and nonnegative functions of the variable x. For brevity, the right-hand side of (38) will be denoted by $F_n(x)$. Let $n \geqslant 2$. Then

$$\begin{aligned} M_{\rho,n-1}(x) &\leqslant (1/3x) \int_x^{4x} M_{\rho,n-1}(t)\,dt \ll (1/x)\, M_{\rho,n}(4x) \ll (1/x)\, F_n(4x) \\ &\ll F_{n-1}(x) + x^{r/2-3/2+n-1} \sum_{x^{1/2}<k\leqslant 2(x)^{1/2}} \frac{k^{2\rho+1} x^{r/2-1/2}}{k^{r-1}} \\ &\ll F_{n-1}(x) + x^{r/2+\rho+n-3/2} \ll F_{n-1}(x), \end{aligned}$$

since ($R_h \ll 1$, $k \leqslant x^{1/2}$)

$$F_{n-1}(x) \gg x^{r/2-1-3/2+n} \sum_{k\leqslant x^{1/2}} k^{2\rho+1} \asymp x^{r/2+\rho+n-3/2}. \tag{40}$$

Hence, it suffices to prove Lemma 7. The starting point will be the expression (13). For the rest of this paragraph let n be sufficiently large, $n = c$, $s = (1/x) + it$ and $s' = (1/x) + it'$. Set

$$H(t, t') = F(s)\, G(s')\, e^{x(s+s')}/s^{\rho+1} s'^{\rho+1}(s + s')^n.$$

Clearly,

$$H(t, t') = \overline{H(-t, -t')} \qquad \text{and} \qquad H(t, t') = H(t', t).$$

Hence, using (13) with $a = 1/x$, we obtain

$$M_{\rho,n}(x) \ll T_1 + T_2 + T_3, \tag{41}$$

where

$$T_1 = \int_{-2w}^{2w} \int_{-2w}^{2w} \cdots dt'\,dt,$$

$$T_2 = \int_{-w}^{w} \left(\int_{2w}^{\infty} \cdots dt' + \int_{-\infty}^{-2w} \cdots dt'\right) dt,$$

and

$$T_3 = \int_w^\infty \int_w^\infty \cdots dt'\,dt + \int_w^\infty \int_{-\infty}^{-2w} \cdots dt'\,dt$$

(all the integrands are $|\,H(t, t')|$). By (19), we obtain for T_1 the estimate

$$T_1 \ll x^{r/2+\rho+n} \int_0^{2w} \left(\int_0^t \frac{x\,dt'}{(1 + x(t - t'))^n}\right) dt \ll x^{r/2+\rho+n-1/2}. \tag{42}$$

Applying the inequality $|ab| \leqslant \frac{1}{2}(|a|^2 + |b|^2)$ to both integrals in T_3, we obtain from the relation $|G(s)| = |F(\bar{s})|$ the estimate

$$T_3 \ll \int_w^\infty \int_w^\infty \frac{|F(s)|^2 + |F(\bar{s})|^2}{(tt')^{\rho+1}((1/x) + |t - t'|)^n} dt' \, dt$$

and by Lemma 6,

$$T_3 \ll x^{n-1} \int_w^\infty \frac{|F(s)|^2}{t^{2(\rho+1)}} dt. \tag{43}$$

Let us estimate T_2. If $|t'| \geqslant 2w$ and $|t| \leqslant w$, then $|s + s'| \asymp |s'| \asymp |t'|$. Hence, by (19),

$$\begin{aligned} T_2 &\ll \int_0^w x^{r/4+\rho/2+1/2} \, dt \int_w^\infty \frac{|F(s')|}{t'^{\rho+1+n}} dt' \\ &\ll x^{r/4+\rho/2} \left(\int_w^\infty \frac{dt'}{t'^{2n}} \int_w^\infty \frac{|F(s')|^2}{t'^{2(\rho+1)}} dt' \right)^{1/2} \\ &\ll x^{r/4+\rho/2+n/2-1/4} \left(\int_w^\infty \frac{|F(s)|^2}{t^{2(\rho+1)}} dt \right)^{1/2}, \end{aligned}$$

i.e.,

$$T_2 \ll x^{(r-1)/4+\rho/2+n/2} \left(\int_w^\infty \frac{|F(s)|^2}{t^{2(\rho+1)}} dt \right)^{1/2}. \tag{44}$$

Thus, we need only an estimate of the integral in (43). If $t \in \mathfrak{B}_{h,k}$, $h > 0$, and $k \leqslant (x)^{1/2}$, then according to (20),

$$|F(s)|^2 \ll \left(\frac{x}{k}\right)^r \frac{\exp\left(-\dfrac{cR_h x}{k^2(1 + x^2(t - (2\pi h/k))^2)}\right)}{(1 + x^2(t - (2\pi h/k))^2)^{r/2}} + \left(\frac{k}{h}\right)^r.$$

Thus, by Lemma 5,

$$\begin{aligned} \int_w^\infty \frac{|F(s)|^2}{t^{2(\rho+1)}} dt &\ll \sum_{h,k} \left(\frac{k}{h}\right)^{2\rho+2} \left(\frac{x}{k}\right)^r \frac{k^{r-1}}{x^{r/2+1/2}} \min{}^{r/2-1/2}\left(\frac{x}{k^2}, \frac{1}{R_h}\right) \\ &\quad + \sum_{h,k} \left(\frac{k}{h}\right)^{2\rho+2+r} \frac{1}{k(x)^{1/2}} \\ &\ll x^{r/2-1/2} \sum_{h,k} \frac{k^{2\rho+1}}{h^{2\rho+2}} \min{}^{r/2-1/2}\left(\frac{x}{k^2}, \frac{1}{R_h}\right), \end{aligned}$$

i.e.,

$$\int_w^\infty \frac{|F(s)|^2}{t^{2(\rho+1)}}\,dt \ll x^{r/2-1/2} \sum_{h,k} \frac{k^{2\rho+1}}{h^{2\rho+2}} \min{}^{r/2-1/2}\left(\frac{x}{k^2}, \frac{1}{R_h}\right), \tag{45}$$

since

$$\sum_{h,k} (k/h)^{2\rho+2+r}\, 1/k(x)^{1/2} \ll x^{-1/2} \sum_{k\leqslant x^{1/2}} k^{2\rho+1+r} \ll x^{r/2+\rho+1/2}$$

and

$$x^{r/2-1/2} \sum_{h,k} (k^{2\rho+1}/h^{2\rho+2}) \min{}^{r/2-1/2}(x/k^2, 1/R_h) \gg x^{r/2-1/2} \sum_{k\leqslant x^{1/2}} k^{2\rho+1}$$
$$\asymp x^{r/2+\rho+1/2}.$$

Using (40)–(45), we get ($F_n(x)$ is the right-hand side of (38))

$$M_{\rho,n}(x) \ll F_n(x) + x^{(r-1)/4+\rho/2+n/2}(x^{1-n}F_n(x))^{1/2} \ll F_n(x),$$

which proves Lemma 7.

4. Ω-Estimates

For a complex s with $\operatorname{Re} s > 0$, we obtain from (9)

$$\int_0^\infty e^{-xs}P_\rho(x)\,dx = \frac{1}{\Gamma(\rho+1)} \sum_{n=0}^\infty a_n \int_{\lambda_n}^\infty e^{-xs}(x-\lambda_n)^\rho\,dx$$
$$- \frac{\pi^{r/2}}{D^{1/2}\Gamma((r/2)+\rho+1)} \int_0^\infty e^{-xs}x^{r/2+\rho}\,dx$$

and thus

$$\int_0^\infty e^{-xs}P_\rho(x)\,dx = F(s)/s^{\rho+1}. \tag{46}$$

This expression was the starting point for Ω-estimates in [11], where only cases corresponding to Theorems A and B from the Introduction were considered. We adapt the method of that paper to our case. We recall that for $\sigma \in (0, 1)$ and $\alpha, \beta \geqslant 0$,

$$\int_0^\infty e^{-x\sigma}x^\beta\,dx = \Gamma(\beta+1)/\sigma^{\beta+1} \tag{47}$$

and

$$\int_0^\infty e^{-x\sigma}x^\beta \log^\alpha(x+1)\,dx \asymp \log^\alpha(1/\sigma)/\sigma^{\beta+1}. \tag{48}$$

THEOREM 4. *Let* α and $\beta > 0$ *be nonnegative real numbers, and let the inequality*

$$0 < P_h \ll h^{-\beta} \log^{-\alpha} h \tag{49}$$

have infinitely many solutions in positive integers h. Then

$$P_\rho(x) = \Omega(x^{(r/2)-1-(\rho+1)/2\beta} \log^{(\rho+1)\alpha/\beta} x).$$

Proof. Our assumptions imply that P_h and R_h are nonzero numbers for all positive h (cf. Lemma 1). By Lemma 1 we choose increasing sequence $h = h_n$, $n = 1, 2,\ldots$ such that the inequality

$$R_h \ll h^{-2\beta} \log^{-2\alpha} h \tag{50}$$

holds, and we suppose that

$$P_\rho(x) = o(x^f \log^g x),$$

where $0 \leqslant f < (r/2) - 1$, $g \geqslant 0$. For $s = \sigma + it$ we obtain from (46) and (48) that

$$F(s)/s^{\rho+1} = o(\log^g \sigma^{-1}/\sigma^{f+1})$$

uniformly in t for σ tending to zero from the right. We put $t = 2\pi h$. According to Lemma 4, we obtain

$$\frac{\pi^{r/2} S_{h,k} e^{-\pi^2 R_h/k^2\sigma}}{D^{1/2} k^r s^{\rho+1} \sigma^{r/2}} - \frac{\pi^{r/2}}{D^{1/2} s^{\rho+1+r/2}} = o\left(\frac{\log^g \sigma^{-1}}{\sigma^{f+1}}\right) + O\left(\frac{e^{-c/k^2\sigma}}{\sigma^{r/2} k^{r/2}}\right) \tag{51}$$

for $\sigma \to 0_+$ uniformly in h. Now we choose $\sigma = R_h/k^2$, $k = 1$ (thus $|s| \asymp h$, $|S_{h,k}| = 1$) and put $h = h_n$, $n = 1, 2,\ldots$. Then $\sigma = \sigma_n$ tends to zero as $n \to \infty$, and we finally obtain

$$R_h\, h^{(\rho+1)/((r/2)-f-1)} \log^{g/((r/2)-f-1)} (1/R_h) \to \infty \tag{52}$$

for $h = h_n$, $n \to \infty$. Now, we consider two cases. First, we suppose that there exists a $\tau = c$ such that

$$R_h \gg h^{-\tau} \tag{53}$$

for all positive integers h. Then $\log(1/R_{h_n}) \asymp \log h_n$, and from (52) it follows that

$$R_h\, h^{(\rho+1)/((r/2)-f-1)} \log^{g/((r/2)-f-1)} h \to \infty$$

for $h = h_n$ as n tends to infinity, but by (50), we obtain

$$R_h h^{2\beta} \log^{2\alpha} h \ll 1.$$

Then either

$$(\rho + 1)/((r/2) - f - 1) > 2\beta, \qquad \text{i.e.,} \qquad f > (r/2) - 1 - ((\rho + 1)/2\beta),$$

or

$$(\rho + 1)/((r/2) - f - 1) = 2\beta > 0 \qquad \text{and} \qquad g/((r/2) - f - 1) > 2\alpha,$$

i.e.,

$$f = (r/2) - 1 - ((\rho + 1)/2\beta) \qquad \text{and} \qquad g > (\rho + 1)(\alpha/\beta).$$

Thus by assumption (53) we proved the assertion of Theorem 4.

If (53) does not hold for any $\tau > 0$, we conclude that the inequality (50) has infinitely many solutions for every $\beta > 0$ with $\alpha = 0$, and we apply the result just proved. We obtain that

$$P_\rho(x) = \Omega(x^{(r/2)-1-\epsilon})$$

for every $\epsilon > 0$, and thus the assertion of Theorem 4 holds in this case also.

We recall the following result (see [11 or 13]).

THEOREM 4^a. *Let all numbers $b_1, b_2, \ldots, b_r$ be rational. Then*

$$P_\rho(x) = \Omega(x^{(r/2)-1}).$$

Proof. We choose a positive integer h such that $R_h = 0$. In (51), we put $f = (r/2) - 1$, $g = 0$, and $k = 1$. Then we obtain that for $\sigma \to 0_+$

$$\pi^{r/2} S_{h,k}/(\sigma + 2\pi i h)^{\rho+1} D^{1/2} k^r = o(1),$$

and this is obviously a contradiction.

THEOREM 5. *Let at least one of the numbers $b_1, b_2, \ldots, b_r$ be irrational. Then*

$$M_\rho(x) = \Omega(x^{r-1-((\rho+1)/\gamma)-\epsilon})$$

for every $\epsilon > 0$, where $\gamma = \gamma(b_1, b_2, \ldots, b_r)$ is defined as in the Introduction.

Proof. For a given positive T, let us write (46) in the form

$$\frac{F(s)}{s^{\rho+1}} = \int_0^T e^{-xs} P_\rho(x)\, dx + \int_T^\infty e^{-xs} P_\rho(x)\, dx.$$

Let $s = \sigma + it$ with $\sigma > 0$. We apply Schwartz's inequality to the first integral, and in the second integral we use the estimate $P_\rho(x) \ll x^\tau$ (this holds, e.g., for $\tau = (r/2) + \rho$). We obtain

$$\frac{F(s)}{s^{\rho+1}} \ll \left(\int_0^T e^{-2x\sigma}\,dx\; M_\rho(T)\right)^{1/2} + \int_T^\infty e^{-x\sigma} x^\tau\,dx,$$

and thus

$$\frac{F(s)}{s^{\rho+1}} \ll \left(\frac{M_\rho(T)}{\sigma}\right)^{1/2} + \frac{e^{-T\sigma/2}}{\sigma^{\tau+1}}. \tag{54}$$

We choose $\beta < \gamma$, $\beta > 0$. Thus, by Lemma 1, the inequality

$$0 < R_h \ll h^{-2\beta} \tag{55}$$

has infinitely many solutions in positive integers h, say, $h = h_n$, $n = 1, 2,\ldots$. In (54) we put $s = \sigma + 2\pi ih$ and $\sigma = R_h$, and apply (21) for $k = 1$. Then

$$R_h^{-r/2} h^{-\rho-1} \ll M^{1/2}(T)/R_h^{1/2} + e^{-TR_h/2}/R_h^{\tau+1}.$$

Let us suppose that

$$M_\rho(x) = o(x^f),$$

where $f < r - 1$ and choose $\epsilon > 0$. Then, for $T \geqslant T_0 = T_0(\epsilon)$ we obtain

$$R_h^{-(r-1)/2} h^{-\rho-1} \ll T^{f/2}(\epsilon + e^{-TR_h/2}/T^{f/2} R_h^{\tau+1/2}). \tag{56}$$

We choose now $T = T_h$ such that

$$e^{-TR_h/2} \leqslant \epsilon T^{f/2} R_h^{\tau+1/2}. \tag{57}$$

Since $e^{-x} \ll x^{-m}$ for every $m = c$ and $x \in [0, \infty)$, we can choose, for example,

$$T_h = (\epsilon c R_h^{-1/2-\tau-m})^{1/(f/2+m)}.$$

Moreover, if

$$\tau + \tfrac{1}{2} < f/2, \tag{58}$$

we can put $T = c/R_h$, and the inequality (57) holds for all $h > h_0$, where h_0 depends on ϵ, f, and τ. From (56) and (57), we obtain

$$R_h^{-(r-1)/2} h^{-\rho-1} \ll \epsilon T^{f/2},$$

and using the expression for $T = T_h$, we obtain (ϵ can be arbitrarily small, $h = h_n \to \infty$)

$$f > r - 1 - ((\rho + 1)/\beta).$$

Similarly to Theorem 4^a, we are able to state

THEOREM 5^a. *Let all the numbers $b_1, b_2, \ldots, b_r$ be rational, and let $P_\rho(x) \ll x^{(r/2)-1}$. Then*

$$M_\rho(x) = \Omega(x^{r-1})$$

Let us recall that the following result always holds:

THEOREM 5^b.

$$M_\rho(x) \gg x^{(r/2)+\rho+(1/2)}.$$

Proof. See [13, p. 69, Theorem 1].

5. O-ESTIMATES

In this paragraph we derive explicit O-estimates from the results of Section 3 and formulate our final results. Because our general formula for O-estimates of $P_\rho(x)$ gives new and non-trivial results only in the case $\rho \leqslant (r/2) - 2$ (see remarks in Section 3), we shall restrict ourselves mainly to this case.

THEOREM 6. *Let $0 \leqslant \rho \leqslant (r/2) - 2$. Then*

$$P_\rho(x) = \Omega(x^{(r-1)/4+\rho/2}), \qquad P_\rho(x) = O(x^{(r/2)-1} \log^\kappa x),$$

where $\kappa = 1$ for $\rho = (r/2) - 2 > 0$, $\kappa = 2$ for $\rho = (r/2) - 2 = 0$ and $\kappa = 0$ otherwise. Further,

$$x^{(r/2)+\rho+(1/2)} \ll M_\rho(x) \ll x^{r-1}$$

for $0 \leqslant \rho < (r/2) - \frac{3}{2}$,

$$x^{r-1} \ll M_\rho(x) \ll x^{r-1} \log x$$

for $\rho = r/2 - \frac{3}{2} \geqslant 0$ and

$$M_\rho(x) \asymp x^{(r/2)+\rho+(1/2)}$$

for $\rho > r/2 - \frac{3}{2}$, $\rho \geqslant 0$.

Proof. The estimates from below follow from Theorem 5^b. Let $\rho = 0$. In (37) we put $z = 1$ and obtain

$$P(x) \ll x^{r/4-1/2} \sum_{h,k} kh^{-2} \min\left(\frac{h}{k}, 1\right)\left(\frac{x}{k^2}\right)^{r/4-1/2}$$

$$\ll x^{r/2-1} \sum_{k \leqslant x^{1/2}} k^{2-r/2}\left(\frac{1}{k}\sum_{h<k}\frac{1}{h} + \sum_{h>k}\frac{1}{h^2}\right)$$

$$\ll x^{r/2-1} \sum_{k \leqslant x^{1/2}} \frac{\log k}{k^{r/2-1}} \ll x^{r/2-1} \log^{\kappa} x.$$

In a similar way, we have from (29)

$$P_\rho(x) \ll x^{r/4-1/2} \sum_{k \leqslant x^{1/2}} k^{\rho}(x/k^2)^{r/4-1/2} \sum_{h=1}^{\infty} (1/h^{\rho+1}) \ll x^{r/2-1} \sum_{k \leqslant x^{1/2}} k^{\rho-r/2+1}$$

for $\rho > 0$, and from (39)

$$M_\rho(x) \ll x^{r/2-1/2} \sum_{k \leqslant x^{1/2}} k^{2\rho+1}(x/k^2)^{r/2-1/2} \sum_{h=1}^{\infty} (1/h^{2\rho+2}) \ll x^{r-1} \sum_{k \leqslant x^{1/2}} k^{2\rho+2-r}$$

for $\rho \geqslant 0$; hence, in both cases, we only need to estimate these sums in an obvious way.

Theorems 4^a, 5^a, and 6 together yield

THEOREM 7. *Let all the numbers $b_1, b_2, \ldots, b_r$ be rational. Then*

$$P_\rho(x) = O(x^{(r/2)-1}) \quad \text{and} \quad P_\rho(x) = \Omega(x^{(r/2)-1})$$

provided $0 \leqslant \rho < (r/2) - 2$,

$$M_\rho(x) = O(x^{r-1}) \quad \text{and} \quad M_\rho(x) = \Omega(x^{r-1})$$

provided $0 \leqslant \rho < r/2 - 3/2$,

$$M_\rho(x) = O(x^{r-1} \log x) \quad \text{and} \quad M_\rho(x) = \Omega(x^{r-1})$$

provided $\rho = r/2 - 3/2 \geqslant 0$, *and finally*

$$M_\rho(x) \asymp x^{(r/2)+\rho+(1/2)} \tag{59}$$

for $\rho > r/2 - 3/2$, $\rho \geqslant 0$.

Let us recall that it follows from the results of [13] and [20] that if the assumptions of Theorem 7 are satisfied, then there are positive constants K depending only on Q, ρ, and b_j such that

$$M_\rho(x) = Kx^{r-1} + o(x^{r-1}) \qquad \text{for } 0 \leqslant \rho < r/2 - 3/2, \tag{60}$$

$$M_\rho(x) = Kx^{r-1} \log x + o(x^{r-1} \log x) \qquad \text{for } 0 \leqslant \rho = r/2 - 3/2, \tag{61}$$

and

$$M_\rho(x) = Kx^{(r/2)+\rho+(1/2)} + O(x^{(r/2)+\rho}) \qquad \text{for } \rho > (r/2) - 1.$$

All the Ω-results for the function $P_\rho(x)$ follow immediately from (60) and (61). Moreover,

$$P_\rho(x) = \Omega(x^{(r/2)-1}(\log x)^{1/2}) \qquad \text{for } \rho = r/2 - 3/2 \geqslant 0.$$

THEOREM 8. *Let at least one of the numbers* $b_1, b_2, \ldots, b_r$ *be irrational. Then*

$$P_\rho(x) = o(x^{(r/2)-1}) \tag{62}$$

provided $0 \leqslant \rho < (r/2) - 2$

$$M_\rho(x) = o(x^{r-1}) \tag{63}$$

provided $0 \leqslant \rho < r/2 - 3/2$.

Proof. First, let $0 \leqslant \rho < r/2 - 3/2$. *Let* $T(x)$ *be the supremum of all* T, *for which*

$$\sum_{1 \leqslant h \leqslant T} (h^{2\rho+2} R_h^{r/2-1/2})^{-1} \leqq x^\alpha,$$

where α is fixed and $0 < \alpha < (r/2) - (3/2) - \rho$. Clearly, $T(x)$ tends to infinity as x tends to infinity. From (39) we obtain

$$\begin{aligned} M_\rho(x) &\ll x^{r/2-1/2} \sum_{k \leqslant x^{1/2}} k^{2\rho+1} \Big(x^\alpha + (x/k^2)^{r/2-1/2} \sum_{h \geqq T} (1/h^{2\rho+2})\Big) \\ &\ll x^{r/2-1/2+\alpha+\rho+1} + T^{-2\rho-1} x^{r-1} = o(x^{r-1}). \end{aligned}$$

In the same way, we obtain (62) from (29) but only for $0 < \rho < (r/2) - 2$.

Finally, let $\rho = 0 < (r/2) - 2$, i.e., $r > 4$. We choose a positive ϵ such that $\epsilon < \min(1/2, (r/2) - 2)$. Put

$$\sum_{0 < h \leqslant n} (h^{1+\epsilon} R_h^{r/4-1/2})^{-1} = \varphi_n$$

$n = 1, 2, \ldots$ Obviously, $\varphi_n < \varphi_{n+1}$. For $t \geqslant 1$, define the function $\varphi(t)$ as follows: $\varphi(t)$ is a continuous linear function in the interval $[n, n+1]$, $\varphi(n) = \varphi_n$, $n = 1, 2, \ldots$ The function $\varphi(t)$ is also continuous and increasing in the interval $[1, \infty)$. Let $\tau(x)$ be defined (obviously uniquely) by the equation

$$x^{(r/4)-1-(\epsilon/2)} = \tau^{\epsilon}(x)\,\varphi(\tau(x)).$$

$\tau(x)$ is a continuous and increasing function, $\lim_{x\to\infty} \tau(x) = \infty$. Finally, we put $z = z(x) = \tau^{-1/2}(x)$ in (37), i.e., $z(x) = o(1)$. Using the inequality $\min(1, k/zh) \leqslant (k/zh)^{\epsilon}$, we obtain from (37)

$$\begin{aligned} P(x) &\ll x^{r/4-1/2} \sum_{h,k} h^{-1} \min(1, k/zh) \min{}^{r/4-1/2}(x/k^2, 1/R_h) + o(x^{r/2-1}) \\ &\ll x^{r/4-1/2} z^{-\epsilon} \sum_{k\leqslant x^{1/2}} k^{\epsilon} \Bigg(\sum_{0<h\leqslant\tau(x)} \frac{1}{h^{1+\epsilon} R_h^{r/4-1/2}} + \frac{x^{r/4-1/2}}{k^{r/2-1}} \sum_{h>\tau(x)} \frac{1}{h^{1+\epsilon}} \Bigg) \\ &\quad + o(x^{r/2-1}) \\ &\ll x^{r/4+\epsilon/2} z^{-\epsilon} \varphi(\tau(x)) + x^{r/2-1} z^{-\epsilon} \tau^{-\epsilon}(x) + o(x^{r/2-1}) = o(x^{r/2-1}). \end{aligned}$$

From the proof we see that our theorem probably cannot be generally strengthened. Actually, using Baire's category method (see, e.g., [12, p. 727]), which was introduced into number theory by Jarník, we could prove the following assertion.

THEOREM 8a. *Let $0 \leqslant \rho < (r/2) - 2$, and let $\varphi(x)$ be a decreasing continuous function, $\varphi(x) = o(1)$. Then there exist a system $b_1, b_2, \ldots, b_r$ such that*

$$P_\rho(x) = o(x^{(r/2)-1}) \qquad \text{and} \qquad P_\rho(x) = \Omega(x^{(r/2)-1}\varphi(x)).$$

An analogous assertion holds for the function $M_\rho(x)$ provided $0 \leqslant \rho < r/2 - \frac{3}{2}$.

In the remainder of this paper, we are going to investigate the dependence of the exact order f_ρ (see (6)) on the arithmetic character of the numbers $b_1, b_2, \ldots, b_r$.

Let at least one of the numbers $b_1, b_2, \ldots, b_r$ be irrational, and let $\gamma = \gamma(b_1, b_2, \ldots, b_r)$ be defined as in the Introduction. According to (7) and (8), we may replace (29), (39), and (37) by

$$P_\rho(x) \ll x^{r/4-1/2} \sum_{k\leqslant x^{1/2}} k^{\rho} \sum_{h=1}^{\infty} h^{-\rho-1} \min{}^{r/2-1}(x^{1/2}/k, 1/P_h), \tag{64}$$

$$M_\rho(x) \ll x^{r/2-1/2} \sum_{k\leqslant x^{1/2}} k^{2\rho+1} \sum_{h=1}^{\infty} h^{-2\rho-2} \min{}^{r-1}(x^{1/2}/k, 1/P_h), \tag{65}$$

and

$$P(x) \ll x^{r/4-1/2} \sum_{k \leqslant x^{1/2}} \sum_{0<h \leqslant x^{r/2}} h^{-1} \min{}^{r/2-1}(x^{1/2}/k,\ 1/P_h). \tag{66}$$

Let us observe that we obtain (66) from (37) putting $z = x^{-((r/4)-1)}$ and recalling that

$$x^{r/4-1/2} \sum_{k \leqslant x^{1/2}} \sum_{0<h \leqslant x^{r/2}} h^{-1} \min{}^{r/2-1}(x^{1/2}/k,\ 1/P_h)$$

$$\gg x^{r/4-1/2} \sum_{k \leqslant x^{1/2}} \sum_{0<h \leqslant x^{1/2}} (1/h) \gg x^{r/4} \log x$$

and

$$(x^{r/4-1/2}/z) \sum_{k \leqslant x^{1/2}} k \sum_{h > x^{r/2}} h^{-2} \min(z(h/k),\ 1) \min{}^{r/2-1}(x^{1/2}/k,\ 1/P_h)$$

$$\ll x^{r/2-3/2} \sum_{k \leqslant x^{1/2}} (x^{r/4-1/2}/k^{r/2-2}) \sum_{h > x^{r/2}} (1/h^2)$$

$$\ll x^{r/4-2} \sum_{k \leqslant x^{1/2}} (1/k^{r/2-2}) \ll x^{r/4}.$$

One can notice that (66) gives in some cases worse results than (37) (but the gap has only a logarithmic character). Because we shall formulate our results in terms of γ, this difference does not matter.

First, we shall state an important auxiliary assertion.

LEMMA 8. *Let α, β, and A be positive real numbers, $A \gg 1$, and let the inequality*

$$P_h \gg h^{-\alpha}$$

hold for all positive integers h. Let $\tau > 1$. Then

$$H_A = \sum_{h=1}^{\infty} h^{-\tau} \min{}^{\beta}(A,\ 1/P_h) \asymp 1 \tag{67}$$

provided $\beta\alpha - \tau < 0$;

$$1 \ll H_A \ll \log A \tag{68}$$

provided $\beta\alpha = \tau$; and

$$H_A \ll A^{\beta-(\tau/\alpha)} \tag{69}$$

provided $\beta\alpha > \tau$.

Further,

$$H(t) = \sum_{0<h\leqslant t} h^{-1} \min{}^{\beta}(A, 1/P_h) \ll A^{\{\beta-(1/\alpha)\}} \log t \tag{70}$$

provided $\beta\alpha \leqslant 1$ *and*

$$H(t) \ll A^{\{\beta-(1/\alpha)\}} \log(t/A^{1/2}) \tag{71}$$

provided $\beta\alpha > 1$ *and* $t^{\alpha} \gg A$. *Here,* $t > c$ *and the symbol* $A^{\{\nu\}}$ *denotes* A^{ν}, $\log A$, 1 *for* $v > 0$, $v = 0$, $v < 0$, *respectively.*

Proof. See [18, pp. 773–774, corollary].

Using these estimates in (64), (65), and (66) and combining the results so obtained with Theorems 4–7, we obtain finally

THEOREM 9. *Let* $\gamma = \gamma(b_1, b_2, \ldots, b_r)$. *Then*

$$\limsup_{x\to\infty} \log M_\rho(x)/\log x = \max(r - 1 - (\rho + 1)/\gamma, (r/2) + \rho + (1/2)).$$

Moreover,

$$M_\rho(x) \asymp x^{(r/2)+\rho+(1/2)}$$

provided $\rho > r/2 - 3/2 - (r - 1)/2(\gamma + 1)$, $\rho \geqslant 0$.

Further,

$$f_\rho = (r/2) - 1 - (1/2\gamma)$$

provided $0 \leqslant \rho \leqslant (r/2) - 2 - (x - 2)/2(\gamma + 1)$ and

$$f_\rho \leqslant r/4 + \rho/2$$

otherwise.

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